

# Finding cores of random 2-SAT formulae via Poisson cloning

Jeong Han Kim<sup>1</sup>

Department of Mathematics and  
Random Graph Research Center  
Seoul, 120-749 Korea  
jehkim@yonsei.ac.kr

**Abstract.** For the random 2-SAT formula  $F(n, p)$ , let  $F_C(n, p)$  be the formula left after the pure literal algorithm applied to  $F(n, p)$  stops. Using the recently developed Poisson cloning model together with the cut-off line algorithm (COLA), we completely analyze the structure of  $F_C(n, p)$ . In particular, it is shown that, for  $\lambda := p(2n-1) = 1 + \sigma$  with  $\sigma \gg n^{-1/3}$ , the core of  $F(n, p)$  has  $\theta_\lambda^2 n + O((\theta_\lambda n)^{1/2})$  variables and  $\theta_\lambda^2 \lambda n + O((\theta_\lambda n)^{1/2})$  clauses, with high probability, where  $\theta_\lambda$  is the larger solution of the equation  $\theta - (1 - e^{-\theta_\lambda \lambda}) = 0$ . We also estimate the probability of  $F(n, p)$  being satisfiable to obtain

$$\Pr[F_2(n, \frac{\lambda}{2n-1}) \text{ is satisfiable}] = \begin{cases} 1 - \frac{1+o(1)}{16\sigma^3 n} & \text{if } \lambda = 1 - \sigma \text{ with } \sigma \gg n^{-1/3} \\ e^{-\Theta(\sigma^3 n)} & \text{if } \lambda = 1 + \sigma \text{ with } \sigma \gg n^{-1/3}, \end{cases}$$

where  $o(1)$  goes to 0 as  $\sigma$  goes to 0. This improves the bounds of Bollobás et al. [8].

## 1 Introduction

An instance of the satisfiability problem is given by a conjunctive normal form (CNF), that is, a conjunction of disjunctions. Each disjunction, or clause, is of the form  $(y_1 \vee \cdots \vee y_k)$ , where  $y_i$ 's are chosen among  $2n$  literals consisting of  $n$  Boolean variables, conditioned that all  $k$  literals are strictly distinct, i.e., no literals with the same underlying variables appear more than once. The problem is whether a given formula has an assignment of truth values (0 or 1) for the  $n$  variables that satisfies the formula. When such an assignment exists, the formula is called satisfiable. It is unsatisfiable, otherwise. It is now well-known that the satisfiability problem is NP-complete ([13]). Even the  $k$ -satisfiability problem, in which each clause consists of exactly  $k$  literals, is known to be NP-complete for  $k \geq 3$  ([13]). In case of  $k = 2$ , there is a polynomial time algorithm [13] to determine whether the instance of the 2-satisfiability problem is satisfiable or not.

The random  $k$ -SAT formula  $F(n, p; k)$  on  $n$  variables is the conjunction of clauses selected with probability  $p$  from the set of  $2^k \binom{n}{k}$  possible clauses, independent of all others. Not surprisingly, the random 2-SAT and the random 3-SAT formulae have been most extensively studied and many research papers regarding the random models have been published. For  $k = 2$ , Chvátal and Reed [12], Goerdt [21] and Fernandez de la Vega [18] independently proved that the random 2-SAT problem undergoes a phase transition at 1, that is,

$$\lim_{n \rightarrow \infty} \Pr[F_2(n, \frac{\lambda}{2n-1}) \text{ is satisfiable}] = \begin{cases} 1 & \text{if } \lambda < 1 \\ 0 & \text{if } \lambda > 1. \end{cases}$$

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Though there is no essential difference, we prefer  $\lambda = p(2n - 1)$  to  $\lambda = 2pn$  because  $p(2n - 1)$  is the mean average degree of each literal. Techniques used to prove the phase transition are essentially based upon the first and the second moment methods for the number of certain structures closely related to the satisfiability. Bollobás et al. [8] took much more sophisticated approaches to determine the scaling window for the problem:

$$\Pr[F_2(n, \frac{\lambda}{2n-1}) \text{ is satisfiable}] = \begin{cases} 1 - \Theta(\frac{1}{\sigma^3 n}) & \text{if } \lambda = 1 - \sigma \text{ with } \sigma \gg n^{-1/3} \\ e^{-\Theta(\sigma^3 n)} & \text{if } \lambda = 1 + \sigma \text{ with } \sigma \gg n^{-1/3}. \end{cases}$$

Though it is believed that the random  $k$ -SAT problem,  $k \geq 3$ , undergoes a similar phase transition, it remains as a conjecture. Only sharp transitions are known due to a seminal result of Friedgut [17]. The upper and lower bounds for the critical value  $\lambda_3$ , (assuming the conjecture is true for  $k = 3$ ) have colorful history. In a series of papers [10, 16, 23, 14, 26, 22, 31, 27, 15], the upper bound of  $\lambda_3$  has been improved to 4.506. There has been considerable work bounding  $\lambda_3$  from below too. The easiest but fundamental algorithm is the pure literal algorithm (PLA). A literal is *pure* in a formula if it belongs to at least one clause of the formula, while its negation is in no clause. The PLA keeps selecting a pure literal, setting it true, and removing clauses containing the literal as they are already satisfied. This procedure may (or may not) yield new pure literals. The algorithm stops when no more pure literal is left. We say that the PLA *succeeds* if no clause remains in the formula after it stops. Clearly, the formula is satisfiable if the PLA succeeds. The converse is not true, for example,  $(y \vee z) \wedge (\bar{y}, \bar{z})$  is satisfiable whereas no pure literal exists.

Broder, Frieze, and Upfal [10] analyzed the PLA for the random 3-SAT problem to show that, if  $\lambda < 1.225$  then the PLA applied to  $F(n, \frac{\lambda}{n^2}; 3)$  succeeds with high probability (whp), and if  $\lambda > 1.275$  then it fails whp. Mitzenmacher [28] used the differential equation method introduced by Wormald [30] to claim that the threshold for the PLA exists and it is the solution of certain equations, which are somewhat complicated. That is, there is  $\lambda(k)$ ,  $k \geq 3$ , so that the PLA applied to  $F(n, \frac{\lambda}{n^{k-1}}; k)$  succeeds whp if  $\lambda < \lambda(k)$ , and fails whp if  $\lambda > \lambda(k)$ . It, however, remains unclear whether it should be regarded as a rigorous proof.

A more advanced algorithm called the unit clause algorithm (UCA) and its variations are analyzed [11, 2, 1, 3] to eventually obtain the lower bound of 3.26. The UCA first chooses a literal uniformly at random and set it true. Then the negation of the literal is removed from the clauses containing it so that they become a clause of length one less. If there are clauses of length 1, or unit clauses, then the UCA chooses a clause uniformly at random among all unit clauses and set the literal in the chosen clause true. The negation of the literal is removed from the clauses containing it. Thus, it is possible that a 0-clause, i.e., a clause without any literal, can be created. The UCA succeeds if no 0-clause is created.

In a recent paper [24], the author introduced the Poisson cloning model  $F_{PC}(n, p; k)$  for random  $k$ -SAT formulae, which is essentially equivalent to the classical model  $F(n, p; k)$  when  $p = \Theta(n^{1-k})$ . That is,

**Theorem 1.1** *Let  $k \geq 2$  and  $p = \Theta(n^{1-k})$ . Then there are constants  $c_1$  and  $c_2$  such that, for any collection  $\mathcal{F}$  of  $k$ -SAT formulae,*

$$c_1 \Pr[F_{PC}(n, p; k) \in \mathcal{F}] \leq \Pr[F(n, p; k) \in \mathcal{F}] \leq c_2 (\Pr[F_{PC}(n, p; k) \in \mathcal{F}]^{\frac{1}{k}} + e^{-n}),$$

where

$$c_1 = k^{1/2} e^{\frac{p}{n} \binom{k}{2} \binom{2n}{k} + \frac{p^2}{2} \binom{2n}{k}} + o(1), \quad c_2 = e^{\frac{p(1-1/k)}{2n} \binom{k}{2} \binom{2n}{k}} \left( \frac{k}{k-1} \right) \left( (k-1)c_1 \right)^{1/k} + o(1).$$

and  $o(1)$  goes to 0 as  $n$  goes infinity.

The cut-off line algorithm (COLA) for the new model is also introduced in a general framework. Using the COLA, one may generate an instance of the Poisson cloning model and simultaneously carry an algorithm such as the PLA. A version of the COLA applied to  $F_{PC}(n, p; k)$  is analyzed to obtain the following result for  $F(n, p; k)$ : Let

$$\lambda(k) := \min_{\rho > 0} \frac{\rho}{(1 - e^{-\rho})^{k-1}},$$

and  $F_C(n, p)$  be the residual formula left after the PRA applied  $F(n, p)$  stops. The residual formula is called the core of  $F(n, p)$ . The set of underlying variables of  $F_C(n, p)$  is denoted by  $C(n, p; k)$ . In other words, a variable is in  $C(n, p; k)$  if and only if a clause of  $F_C(n, p; k)$  contains it.

**Theorem 1.2** *Let  $\lambda(n, p; k) = p \binom{2n-1}{k-1}$ ,  $k \geq 3$  and  $\sigma \gg n^{-1/2}$ . Supercritical Phase: If  $\lambda(n, p; k) < \lambda(k) - \sigma$  is uniformly bounded from below by 0 and  $i_0(k)$  is the minimum  $i$  such that  $2^k \binom{i}{k} \geq 2i/k$ , then*

$$\Pr[C(n, p; k) \neq \emptyset] \leq 2e^{-\Omega(\sigma^2 n)} + O(n^{-(1-2/k)i_0(k)}).$$

*Supercritical Phase: If  $\lambda := \lambda(n, p; k) = \lambda(k) + \sigma$  is uniformly bounded from above, then, for the largest solution  $\theta_\lambda$  of the equation  $\theta^{\frac{1}{k-1}} - 1 + e^{-\theta\lambda} = 0$  and all  $\alpha$  in the range  $1 \ll \alpha \ll \sigma n^{1/2}$ ,*

$$\Pr[|C(n, p; k)| - \theta_\lambda^{\frac{2}{k-1}} n \geq \alpha(n/\sigma)^{1/2}] = e^{-\Omega(\alpha^2)}.$$

*In particular, the PRA succeeds with high probability if  $\lambda(n, p; k) = \lambda(k) - \sigma$  with  $\sigma \gg n^{1/2}$ , and it does not succeed with high probability if  $\lambda(n, p; k) = \lambda(k) + \sigma$  with  $\sigma \gg n^{1/2}$ .*

Most of structural properties of the core can be found in [24] too. The Poisson cloning model and the cut-off line algorithm will be presented in detail in the next section .

For  $k = 2$ , the PLA may not succeed with nontrivial probability even for  $\lambda_p := p(2n-1) < 1$ . For example, there could be a pair of clauses  $(y \vee z)$  and  $(\bar{y} \vee \bar{z})$  for two variables  $y$  and  $z$  with non-trivial probability. Hence, we may expect, at best, that if  $\lambda_p := p(2n-1) < 1$  then  $F_C(n, p) := F_C(n, p; 2)$  consists of variables of type  $(1, 1)$  only. Here and in general, a variable  $x$  is of type  $(i, j)$  in a formula if  $x$  appears in  $i$  clauses and  $\bar{x}$  appears in  $j$  clauses of the formula. The type of a literal  $\bar{x}$  is determined by the type of  $x$ . Taking similar approaches used to analyze the structure of the core of the random digraph [25], we will actually prove it and, in case that  $\lambda_p > 1$ , we prove that  $F_C(n, p)$  has many variables of type larger  $(1, 1)$  and the formula is not satisfiable whp. All the proofs presented here do not depend on [25] though.

Other interesting properties for  $F_C(n, p)$  are studied too. Denoted by  $C_{n,p}(i, j)$  is the set of all variables of type  $(i, j)$  in  $F_C(n, p)$  and  $C_{n,p} = \cup_{(i,j) \geq (1,1)} C_{n,p}(i, j)$  is the set of underlying variables of  $F_C(n, p)$ . Due to the following lemma, the structure of the core  $F_C(n, p)$  can be well understood provided tight upper and lower bounds for  $|C_{n,p}(i, j)|$ 's are found,  $(i, j) \geq (1, 1)$ .

**Theorem 1.3** *Suppose two formulae have the same number of clauses on the same number of underlying variables, and all underlying variables are of type at least  $(1, 1)$ . Then the two formulae are equally likely to be the core of  $F(n, p)$ .*

The proof of the theorem is not difficult and presented in Section 4.

For variables  $x$  of type  $(1, 1)$  in  $F_C(n, p)$ , the conjunction  $(x \vee y) \wedge (\bar{x} \vee z)$  of two clauses containing  $x$  and  $\bar{x}$  may be replaced by  $(y \vee z)$ . The replacement is called a resolution of  $x$ . It is clear that the satisfiability is not affected by a series of such resolutions. The formula obtained after all possible resolutions of type  $(1, 1)$  variables is called the kernel of  $F(n, p)$  and denoted by  $F_K(n, p)$ . It is

worth to notice that clauses in  $F_K(n, p)$  may not consist of strictly distinct literals. Clearly, all variables of  $F_K(n, p)$  are of type larger than  $(1, 1)$ , counting a loop  $(x \vee x)$  twice in the degree of  $x$ .

Let  $D_{n,p}(i, j) = \sup_{(i', j') \geq (i, j)} C_{n,p}(i', j')$ . Then  $D_{n,p}(1, 1) = C_{n,p}$  and  $K_{n,p} := D_{n,p}(2, 1) \cup D_{n,p}(1, 2)$  is the set of underlying variables of  $F_K(n, p)$ . When  $C_{n,p}(i', j')$  are all small for  $(i', j') \geq (i, j)$ , it is sometimes more useful and/or easier to bound the size of  $D_{n,p}(i, j)$  rather than individual  $C_{n,p}(i', j')$ . We also denote  $M_{n,p}(i, j)$  to be the sum of degrees of all variables in  $D_{n,p}(i, j)$  and their negations, where the degree  $d(y)$  of a literal  $y$  is the number of clauses containing it. Clearly,

$$M_{n,p}(i, j) = \sum_{(i', j') \geq (i, j)} (i' + j') |C_{n,p}(i', j')|.$$

Notice that the numbers of clauses in  $F_C(n, p)$  and  $F_K(n, p)$  are  $\frac{1}{2}M_{n,p}(1, 1)$  and  $\frac{1}{2}(M_{n,p}(1, 2) + M_{n,p}(2, 1) - M_{n,p}(2, 2))$ , respectively. Finally, we set

$$P_\ell(\mu) = \Pr[\text{Poi}(\mu) = \ell] = e^{-\mu} \frac{\mu^\ell}{\ell!}, \quad \text{and} \quad Q_\ell(\mu) = \Pr[\text{Poi}(\mu) \geq \ell] = e^{-\mu} \sum_{\ell' \geq \ell} \frac{\mu^{\ell'}}{\ell'!}.$$

In statements in theorems, lemmas and corollaries of this paper, we use the following convention.

**Convention:** When we say that a statement is true for all  $\alpha$  in the range  $a \ll \alpha \ll b$ , it actually means that there is (small) constant  $\varepsilon > 0$  so that the statement is true for  $\alpha$  in the range  $a/\varepsilon \leq \alpha \leq \varepsilon b$ .

**Theorem 1.4** *Suppose  $p(2n - 1) = 1 + \sigma$  is uniformly bounded from above with  $\sigma \gg n^{-1/3}$ . Let  $\lambda = 1 + \sigma$ ,  $\Delta > 0$  and  $1 \ll \alpha \ll (\theta_\lambda n)^{1/2}$ . Then, for fixed  $(i, j)$  and  $P_i = P_i(\theta_\lambda \lambda)$  and  $Q_i = Q_i(\theta_\lambda \lambda)$ ,*

$$\Pr \left[ \left| |C_{n,p}(i, j)| - P_i P_j n \right| \geq \alpha \theta_\lambda^{i+j-3} (\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\theta_\lambda^{i+j-3} \alpha^2)} + e^{-\Omega(\alpha^2)},$$

and, assuming  $i \geq j$ ,

$$\Pr \left[ \left| |D_{n,p}(i, j) \cup D_{n,p}(j, i)| - (2Q_i Q_j - Q_i Q_i) n \right| \geq \alpha \theta_\lambda^{i+j-3} (\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\theta_\lambda^{i+j-3} \alpha^2)} + e^{-\Omega(\alpha^2)},$$

and

$$\Pr \left[ \left| M_{n,p}(i, j) - \theta_\lambda \lambda (Q_{i-1} Q_j + Q_i Q_{j-1}) n \right| \geq \alpha \theta_\lambda^{i+j-3} (\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\theta_\lambda^{i+j-3} \alpha^2)} + e^{-\Omega(\alpha^2)}.$$

Moreover,

$$\Pr[|D_{n,p}(i, j)| \geq \ell] \leq O\left(\frac{((1 + \frac{\alpha}{(\theta_\lambda n)^{1/2}}) \theta_\lambda \lambda)^{(i+j)\ell/2}}{(\ell!)^{1/2}}\right) + e^{-\Omega(\alpha^2)}.$$

A stronger theorem (Theorem 3.3, see also Main Lemma in Section 3) is to be first proved and Theorem 1.4 will follow as a corollary. Bounds for the sizes of the core and the kernel may be obtained from Theorem 1.4. Estimations for  $|F_C(n, p)|, |F_K(n, p)|$  are possible too, where, in general,  $|F|$  is the number of clauses in the formula  $F$ .

**Corollary 1.5** *For the core  $F_C(n, p)$  of  $F(n, p)$  and the set  $C_{n,p}$  of underlying variables of the core,*

$$\Pr \left[ \left| |C_{n,p}| - \theta_\lambda^2 n \right| \geq \alpha (\theta_\lambda n)^{1/2} \right] \leq e^{-\Omega(\alpha^2)},$$

and

$$\Pr \left[ \left| |F_C(n, p)| - \theta_\lambda^2 \lambda n \right| \geq \alpha(\theta_\lambda n)^{1/2} \right] \leq e^{-\Omega(\alpha^2)}.$$

For the kernel  $F_K(n, p)$  of  $F(n, p)$  and the set  $K_{n, p}$  of underlying variables of the kernel,

$$\Pr \left[ \left| |K_{n, p}| - \theta_\lambda^2 (1 - \lambda^2 e^{-2\theta_\lambda \lambda}) n \right| \geq \alpha(\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\alpha^2)},$$

$$\Pr \left[ \left| |F_K(n, p)| - \theta_\lambda^2 \lambda (1 - \lambda e^{-2\theta_\lambda \lambda}) n \right| \geq \alpha(\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\alpha^2)}.$$

In brief, we may also have

**Corollary 1.6** *Let  $\lambda = 1 + \sigma$  with  $n^{-1/3} \ll \sigma < 1$ . Then, with high probability, the pure literal algorithm applied to  $F(n, \frac{\lambda}{2n-1})$  stops leaving  $\Theta(\sigma^2 n)$  type (1, 1) variables,  $\Theta(\sigma^3 n)$  type (2, 1) or (1, 2) variables, and  $O(\sigma^4 n)$  clauses containing other type variables. Moreover, once  $C_{n, p}(i, j)$ ,  $(i, j) \geq (1, 1)$ , are given, the residual formula is the uniform random formula conditioned on  $C_{n, p}(i, j)$ .*

The analysis of the structure of the core yields almost optimal bounds for the probability of satisfiability, improving bounds of Bollobás et. al. [8].

**Theorem 1.7** *If  $\lambda_p = 1 - \sigma$  is uniformly bounded from below by 0 with  $n^{-1/3} \ll \sigma \ll 1$ , then, with probability  $1 - \frac{15+o(1)}{16\sigma^3 n}$ , all the variables in  $C(n, p)$  are of type (1, 1). That is,*

$$\Pr \left[ K(n, p) = \emptyset \right] = 1 - \frac{15 + o(1)}{16\sigma^3 n}. \quad (1.1)$$

In particular,  $\Pr[K(n, p) = \emptyset] = 1 - O((\sigma^3 n)^{-1})$  for all  $\sigma$  in the range  $n^{-1/3} \ll \sigma < 1$ . We also have

$$\Pr[F(n, p) \text{ is satisfiable}] = 1 - \frac{1 + o(1)}{16\sigma^3 n}.$$

**Theorem 1.8** *If  $\lambda_p = 1 + \sigma$  is uniformly bounded from above, then  $F(n, p)$  is unsatisfiable with probability  $1 - e^{-\Theta(\sigma^3 n)}$ , i.e.,*

$$\Pr[F(n, p) \text{ is satisfiable}] = e^{-\Theta(\sigma^3 n)}.$$

In the next section, we present the Poisson cloning model and the cut-off line algorithm together with an useful large deviation inequality called generalized Chernoff bound. Then, Theorem 1.4 and Corollaries 1.5 and 1.6 will be proven in Section 3. Section 4 is for the proofs of Theorems 1.3 1.7 and 1.8.

## 2 Poisson Cloning Model and Cut-Off Line Algorithm

**Poisson Cloning Model:** The Poisson cloning model is partially motivated by the fact that the degree  $d(y)$  of a literal  $y$  in  $F(n, p)$  is the binomial distribution  $\text{Bin}(2n - 1, p)$ , which is close to  $\text{Poi}(p(2n - 1))$  when  $p = \Theta(n^{-1})$ . Here the degree  $d(y)$  of  $y$  is the number of clauses in  $F(n, p)$  containing  $y$  and

$$\Pr[\text{Bin}(2n - 1, p) = \ell] = \binom{2n - 1}{\ell} p^\ell (1 - p)^{2n - 1 - \ell}, \quad \Pr[\text{Poi}(\lambda) = \ell] = e^{-\lambda} \frac{\lambda^\ell}{\ell!}.$$

Though the degrees  $d(y)$ 's are not exactly independent, they are expected to behave like i.i.d random variables. Thus, it has been desirable to introduce a new model for the random  $k$ -SAT formulae in which the degrees are i.i.d Poisson random variables. Inspired by the configuration model for random regular graphs, see e.g. [5], [6], [7], and [29], the author have introduced the Poisson cloning model with the desired properties and show that the new model is not much different from the classical model in the sense of Theorem 1.1.

To analyze various properties of random graphs and random SAT formulae such as cores and giant components, the cut-off line algorithm is introduced too. In this section, we present the Poisson cloning model and the cut-off line algorithm, and related lemmas as well as a large deviation inequality called generalized Chernoff bound.

For a new random 2-SAT model  $F_{PC}(n, p; 2)$ , we take i.i.d Poisson  $\lambda := p(2n - 1)$  random variables  $d_y$  for each  $y$  in the set  $Y$  of all literals, and then take  $d_y$  copies of each  $y$ . The copies of a literal  $y$  are called *clones of  $y$* , or simply  *$y$ -clones*. Since the sum of Poisson random variables is also Poisson, the total number  $N_\lambda := \sum_{y \in Y} d_y$  of clones is a Poisson  $2\lambda n$  random variable. It is sometimes convenient to take a reverse, but equivalent, construction. We first take a Poisson  $2\lambda n$  random variables  $N_\lambda$  and then take  $N_\lambda$  unlabelled clones. Each clone is independently labelled as  $y$ -clone uniformly at random, in the sense that  $y$  is chosen uniformly at random from  $Y$ . It is well-known that the numbers  $d_y$  of  $y$ -clones are i.i.d Poisson  $\lambda$  random variables.

If  $N_\lambda$  is even, the formula  $F_{PC}(n, p; 2)$  is to be defined by generating a (uniform) random perfect matching on those  $N_\lambda$  clones and contracting clones of a literal  $y$  into  $y$ . That is, an edge consisting of a  $y$ -clone and a  $z$ -clone in the perfect matching yields the clause  $(y \vee z)$  in  $F_{PC}(n, p; 2)$  with multiplicity. If  $y = z$ , it produces, a loop  $(y \vee y)$ , which contributes 2 in the degree of  $y$ . It turns out that there are many ways to generate the random perfect matchings and we may choose one that makes given problems easier to analyze. Some specific ways will be discussed when the cut-off line algorithm is introduced.

If  $N_\lambda$  is odd, we arbitrarily choose a clone, say  $y$ -clone. This clone induces a 1-clause, called a defected clause, consisting of  $y$ . The defected clause contribute only 1 to the degree of the corresponding literal. The same procedure taken for the case of even  $N_\lambda$  are to be carried for the rest of clones. Strictly speaking  $F_{PC}(n, p; 2)$  varies depending on how to construct the defected clause. However, for any collection  $\mathcal{F}$  of 2-SAT formulae, the probability that  $F_{PC}(n, p; 2)$  is in  $\mathcal{F}$  does not depend on how the defected clause is chosen (for odd  $N_\lambda$ ), since  $F_{PC}(n, p; 2) \notin \mathcal{F}$  whenever there is a non-standard clause in  $F_{PC}(n, p; 2)$ . Thus it is normally unnecessary to describe  $F_{PC}(n, p; 2)$  for odd  $N_\lambda$ . For  $k \geq 3$ , the Poisson cloning model  $F_{PC}(n, p; k)$  for random  $k$ -SAT problems may be similarly defined.

Theorem 1.1 has been proved using somewhat straightforward computations for  $\Pr[F(n, p; k) = F]$  and  $\Pr[F_{PC}(n, p; k) = F]$ .

**Cut-Off Line Algorithm (COLA):** To generate a uniform random perfect matching on  $N_\lambda$  clones, we may keep matching two unmatched clones uniformly at random. Another way is to choose the first clone as we like and match it to a clone chosen uniformly at random among all other unmatched clones. Clearly, there are many ways to choose the first clone. This is a big advantage since we may select a way that makes the given problem easier to analyze. In general, a sequence of choice functions will tell how to choose the first clone at each step. A choice function may be deterministic or random. If  $N_\lambda$  is even, this would yield a uniform perfect matching regardless what the choice functions are. If only one clone, say of  $y$ , remains unmatched, we just add the defected clause consisting of  $y$ .

It is useful to introduce a more specific way to choose the second clone uniformly at random. The way presented here will be useful to analyze some algorithms like the PRA. First, we independently

assign, to each clone, a uniform random real number between 0 and  $\lambda$ . For the sake of convenience, we say that a clone is the largest, smallest, etc. if so is its assigned number. Each choice function is to choose an unmatched clone without changing the (joint) distribution of the numbers assigned to all other unmatched clones. A choice function satisfying this condition is called *oblivious*. For instance, a choice function is oblivious if it chooses a clone of a pure literal. If a choice function chooses a largest  $v$ -clone, it is not oblivious, as it changes the distribution of the numbers assigned to other unmatched  $v$ -clones.

Once an unmatched clone is chosen by an oblivious choice function, the largest clone among all other unmatched clones are to be matched to the chosen clone. This may be further implemented using the Poisson  $\lambda$ -cell: First, map a  $y_j$ -clone with assigned number  $r$  to the point  $(r, j)$  in the two dimensional plane. One may think that there are  $2n$  horizontal line segments in  $\mathbf{R}^2$  from  $(0, j)$  to  $(\lambda, j)$ ,  $j = 1, \dots, 2n$  and, on each line segment, there are i.i.d. uniform  $d_{y_j}$  points that tell the assigned numbers for  $d_{y_j}$  clones of  $y_j$ . This rectangular configuration is called a *Poisson  $\lambda$ -cell*. Each line segment of the Poisson  $\lambda$ -cell with the points is an independent Poisson arrival process with density 1, up to time  $\lambda$ .

The cut-off line algorithm (COLA) can be described as follows. Initially, the cut-off line is the vertical line in  $\mathbf{R}^2$  containing the point  $(\lambda, 0)$ . At the first step, once the oblivious choice function chooses a clone, we move the cut-off line to the left until a clone is on the line. The clone is clearly the largest unmatched clone, excluding the chosen clone. The new cut-off value, denoted by  $\Lambda_1$ , is the assigned number to the clone. The new cut-off line is, of course, the vertical line containing  $(\Lambda_1, 0)$ . Keep repeating this procedure, one may obtain the  $i^{\text{th}}$  cut-off value  $\Lambda_i$  and the corresponding cut-off line. It is crucial to note that, provided all choice functions are oblivious, once  $\Lambda_i$  is given then all numbers assigned to unmatched clones are i.i.d uniform random numbers between 0 to  $\Lambda_i$ .

For  $\theta$  in the range  $0 \leq \theta \leq 1$ , let  $\Lambda(\theta)$  be the cut-off value when  $(1 - \theta^2)\lambda n$  or more clones are matched for the first time. Conversely, let  $N(\theta)$  be the number of matched clones until the cut-off line reaches  $\theta\lambda$ . Two versions of the cut-off line lemma have been proven in [24].

**Lemma 2.1** (*Cut-off Line Lemma*) *Let  $\lambda > 0$  be fixed. Then, for  $\theta_1 < 1$  uniformly bounded below from 0 and  $0 < \Delta \leq n$ ,*

$$\Pr \left[ \max_{\theta: \theta_1 \leq \theta \leq 1} |\Lambda(\theta) - \theta\lambda| \geq \frac{\Delta}{n} \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)^n}\})},$$

and

$$\Pr \left[ \max_{\theta: \theta_1 \leq \theta \leq 1} |N(\theta) - 2(1 - \theta^2)\lambda n| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)^n}\})}.$$

For the Poisson  $\lambda$ -cell conditioned on  $N_\lambda = N$ , a similar lemma may be obtained.

**Lemma 2.2** (*Cut-off Line Lemma for  $N$  clones*) *Let  $k \geq 2$ ,  $\lambda > 0$  be fixed. Then, for the Poisson  $\lambda$ -cell conditioned on  $N_\lambda = N$ , and for  $\theta_1 < 1$  uniformly bounded below from 0 and  $0 < \Delta \leq N$ ,*

$$\Pr \left[ \max_{\theta: \theta_1 \leq \theta \leq 1} |\Lambda(\theta) - \theta\lambda| \geq \frac{\Delta}{N} \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)^N}\})},$$

and

$$\Pr \left[ \max_{\theta: \theta_1 \leq \theta \leq 1} |N(\theta) - (1 - \theta^2)N| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)^N}\})}.$$

For the proof of the cut-off line lemma, a large deviation inequality, called generalized Chernoff bound, has been used. Here, we present a version of it that is useful for our analysis. A proof can be found in [24].

**Lemma 2.3** (*Generalized Chernoff bound*) Let  $X_1, \dots, X_m$  be a sequence of random variables. Suppose

$$E[X_i | X_1, \dots, X_{i-1}] \leq \mu_i, \quad (2.1)$$

and there are  $a_i, b_i$  and  $\xi_0$  so that

$$E[(X_i - \mu_i)^2 | X_1, \dots, X_{i-1}] \leq a_i, \quad (2.2)$$

and

$$E[(X_i - \mu_i)^3 e^{\xi(X_i - \mu_i)} | X_1, \dots, X_{i-1}] \leq b_i \quad \text{for all } 0 \leq \xi \leq \xi_0. \quad (2.3)$$

If  $\delta \xi_0 \sum_{i=1}^m b_i \leq \sum_{i=1}^m a_i$  for some  $0 < \delta \leq 1$ , then

$$\Pr \left[ \sum_{i=1}^m X_i \geq \sum_{i=1}^m \mu_i + \Delta \right] \leq e^{-\frac{1}{3} \min\{\delta \xi_0 \Delta, \Delta^2 / \sum_{i=1}^m a_i\}},$$

for all  $\Delta > 0$ . Furthermore, if  $X_1, \dots, X_m$  are independent and satisfy (2.2) for  $\mu_i = E[X_i]$  and

$$\left| E[(X_i - E[X_i])^3 e^{\xi(X_i - E[X_i])}] \right| \leq b_i \quad \text{for all } \xi \text{ in the range } |\xi| \leq \xi_0, \quad (2.4)$$

then  $\delta \xi_0 \sum_{i=1}^m b_i \leq \sum_{i=1}^m a_i$  for  $0 < \delta \leq 1$  implies that

$$\Pr \left[ \left| \sum_{i=1}^m X_i - \sum_{i=1}^m E[X_i] \right| \geq \Delta \right] \leq e^{-\frac{1}{3} \min\{\delta \xi_0 \Delta, \frac{\Delta^2}{\sum_{i=1}^m a_i}\}},$$

for all  $\Delta > 0$ .

We conclude this section by presenting a corollary that can be applied to random walks with negative drift.

**Corollary 2.4** Suppose (2.1)-(2.3) hold with  $\mu_i = -h$  for a constant  $\beta > 0$ . If  $\delta \xi_0 \sum b_i \leq \sum a_i$  for some  $0 < \delta \leq 1$ , then

$$\Pr \left[ \sum_{i=1}^m X_i \geq \Delta \right] \leq e^{-\Omega(\min\{\delta \xi_0 (\Delta + hm), (\Delta + hm)^2 / \sum_{i=1}^m a_i\})}.$$

### 3 Pure literal algorithm for the random 2-SAT problem

As mentioned in the previous section, the COLA is useful to realize some algorithms like the PLA. The following specific COLA is used to analyze the structure of the core of  $F_{PC}(n, p)$ .

**COLA (for core):** Construct a Poisson  $\lambda$ -cell. If a variable is of types  $(0, i)$  or  $(i, 0)$ , put all clones of it and its negation into a stack in an arbitrary order. This does not mean that the clones are removed from the  $\lambda$ -cell.

(a) If the stack is empty, go to (b). If the stack is nonempty, choose the first clone in the stack and move the cut-off line to the left until the largest unmatched clone, excluding the chosen clone, is found. (The stack naturally defines choice functions.) Then, match the largest unmatched clone



to the chosen clone. Remove all matched clones from the stack and from the cell. If there are new variables of type  $(0, i)$  or  $(i, 0)$ , then put all clones of them and their negations in the stack. Repeat (a).

(b) Choose a clone uniformly at random from all unmatched clones and put it in the stack. Then, go to (a).

The steps carried by the instruction described in (b) are called *free steps* as it is free to choose any clone. We will call unmatched clones of pure literal *light* and the other unmatched clones *heavy*. A literal is called heavy if it not pure.

According to the cut-off line lemma, one may expect that there are  $2\theta^2\lambda n$  unmatched clones (when the cut-off line is) at  $\theta\lambda$ . The number of heavy clones at  $\theta\lambda$  is expected to be close to  $2(1 - e^{-\theta\lambda})\theta\lambda n = \Theta(\theta^2 n)$ . (See (3.3) below.) Thus, the number of light clones seems to be close to

$$2\theta^2\lambda n - 2(1 - e^{-\theta\lambda})n = 2\theta\lambda n(\theta - 1 + e^{-\theta\lambda}),$$

which is  $\Theta(\theta^3 n)$  provided  $\theta \gg |\lambda - 1|$ . If  $\theta$  is small, however, this observation would give us no information. This is due to the fact that the standard deviation for the number of heavy clones is  $\theta n^{1/2}$  so that, for  $\theta^3 n \ll \theta n^{1/2}$ , or  $\theta \ll n^{-1/4}$ , it is unclear whether the number of light clones is positive or not.

A more careful analysis starts from the observation that, when  $\theta$  is small, most of heavy variables are of type  $(1, 1)$  and that the two clauses containing such a variable and its negation may be resolved to one clause. In other words, the two clause  $(x \vee y)$  and  $(\bar{x} \vee z)$  may be replaced by  $(y \vee z)$ , which is called a resolution. After a series of such resolutions, all variables of type  $(1, 1)$  may disappear.

To take an advantage of this fact, we will introduce many phases. Let  $\frac{1-\theta\lambda}{10} \leq \beta \leq \frac{1-\theta\lambda}{2}$ . The first phase starts at the beginning of the whole process. For  $j \geq 1$ , the  $j^{\text{th}}$  phase ends and the  $(j+1)^{\text{th}}$  phase begins when the cut-off line reaches  $(1 - \beta)^j \lambda$ . At the beginning of each phase, all variables of type  $(1, 1)$  and their unmatched clones are called passive. All other unmatched clones are called active. These terms do not change until the beginning of the next phase. So, variables that become type  $(1, 1)$  only after the current phase starts remain active until the end of the phase. Once a clone becomes pure, it plays the same role regardless of being passive or active. The procedure (b) of COLA also need to be replaced by

(b)\* Choose a clone uniformly at random from all unmatched active clones and put it in the stack. If there is no active clone, stop. Otherwise, go to (a).

As a stack is used, if one of the two unmatched clones of a passive variable and its negation were matched in a step then the choice function in the next step must choose the other clone. Thus, the situation is exactly the same except the number of passive variables decreases by 1. This means that the COLA applied without passive clones is essentially the same as the original algorithm. In this sense, we may say that two active clones are matched if so are they after the resolutions of matched passive clones. Here the resolution has the natural meaning: Two edges  $\{z_1, z_2\}, \{z_3, z_4\}$  with clones  $z_2, z_3$  of a passive variable and its negation is reduced to the one edge  $\{z_1, z_4\}$ . Conversely, an active clone may be regarded as unmatched if it is not matched or it is not matched after the resolutions.

Let  $\Lambda_C$  be the cut-off value when no light clone remains for the first time in the COLA applied to the Poisson  $\lambda$ -cell. The main lemma shows that  $\lambda_C$  is highly concentrated near  $\theta_\lambda \lambda$ , as expected, with standard deviation  $(\theta_\lambda n)^{-1/2}$ . Once  $\Lambda_C$  is determined, the unmatched clones form the Poisson  $\Lambda_C$ -cell without pure literals.

**Lemma 3.1** (Main Lemma) *Let  $\lambda = 1 + \sigma$  with  $\sigma \gg n^{-1/3}$ . Then, for all  $\alpha$  with  $1 \ll \alpha \ll (\theta_\lambda^3 n)^{1/2}$ ,*

$$\Pr[|\Lambda_C - \theta_\lambda \lambda| \geq \alpha(\theta_\lambda n)^{-1/2}] = e^{-\Omega(\alpha^2)}.$$

For the proof, we first estimate the number of active clones at the beginning of each phase. Let  $N_j$  be the number of active clones at the beginning of the  $j^{\text{th}}$  phase and let  $M_j$  be the number of matched active clones during the entire  $j^{\text{th}}$  phase. Then, the cut-off line lemma for  $N_j$  clones, or Theorem 2.2, gives

$$\Pr[|M_j - (1 - (1 - \beta)^2)N_j| \geq \Delta|N_j] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{N_j}\})}. \quad (3.1)$$

Notice that the number  $N_{j+1}$  of active clones at the beginning of the next phase is  $N_j - M_j - 2B_j$ , where  $B_j$  is the number of variables of type  $(1, 1)$  at  $(1 - \beta)^j \lambda$  that were of type larger than  $(1, 1)$  at  $(1 - \beta)^{j-1} \lambda$ . (Recall that an active clone is regarded as unmatched if it is not matched or it is not matched after resolutions.) For a literal  $y$  and  $0 \leq \theta < \theta' \leq 1$ , denoted by  $d_y(\theta, \theta')$  is the number of  $y$ -clones larger than or equal to  $\theta \lambda$  and smaller than  $\theta' \lambda$ , and  $d_y(\theta) = d_y(0, \theta)$ . Then, for  $\theta_j = (1 - \beta)^{j-1}$ ,

$$B_j = \sum_{x \in X} 1(d_x(\theta_{j+1}) = d_{\bar{x}}(\theta_{j+1}) = 1) 1(d_x(\theta_{j+1}, \theta_j) + d_{\bar{x}}(\theta_{j+1}, \theta_j) \geq 1).$$

Observe that  $(d_x(\theta_{j+1}), d_{\bar{x}}(\theta_{j+1}), d_x(\theta_{j+1}, \theta_j), d_{\bar{x}}(\theta_{j+1}, \theta_j))$ ,  $x \in X$ , are i.i.d 4-tuples of independent Poisson random variables with means  $\theta_{j+1} \lambda$ ,  $\theta_{j+1} \lambda$ ,  $(\theta_j - \theta_{j+1}) \lambda$ ,  $(\theta_j - \theta_{j+1}) \lambda$ , respectively. Applying the generalized Chernoff bound, we have

$$\Pr[|B_j - (\theta_{j+1} \lambda)^2 e^{-2\theta_{j+1} \lambda} (1 - e^{-2\theta_j \lambda}) n| \geq \Delta] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_j^3 n}\})}. \quad (3.2)$$

Therefore,  $N_{j+1}$  is expected to be close to  $(1 - \beta)^2 N_j - 2(\theta_{j+1} \lambda)^2 e^{-2\theta_{j+1} \lambda} (1 - e^{-2\theta_j \lambda}) n$ . Applying this inductively, we expect that  $N_j$  is close to  $2\theta_j^2 \lambda (1 - \lambda e^{-2\theta_j \lambda}) n$ .

Let

$$\begin{aligned} H_j &= \sum_{x \in X} (d_x(\theta_j) + d_{\bar{x}}(\theta_j)) 1\left((d_x(\theta_j), d_{\bar{x}}(\theta_j)) > (1, 1)\right) \\ &= \sum_{x \in X} (d_x(\theta_j) + d_{\bar{x}}(\theta_j)) 1\left((d_x(\theta_j), d_{\bar{x}}(\theta_j)) \geq (1, 1)\right) - 2 \sum_{x \in X} 1\left(d_x(\theta_j) = d_{\bar{x}}(\theta_j) = 1\right). \end{aligned}$$

Then  $H_j$  is the number of active heavy clones at the beginning of the  $j^{\text{th}}$  phase unless there is a free step before  $\theta_j \lambda$ . Generally,  $H_j$  is an upper bound for the number of heavy clones and  $L_j := N_j - H_j$  is a lower bound for the number of light clones. The bounds may be strict only when there is a free step before the cut-off line reaches  $\theta_j \lambda$ .

As  $(d_x(\theta_j), d_{\bar{x}}(\theta_j))$  are i.i.d pairs of independent Poisson random variables with mean  $\theta_j \lambda$ , the generalized Chernoff bound gives

$$\Pr\left[|H_j - 2\theta_j \lambda (1 - e^{-\theta_j \lambda} - \theta_j \lambda e^{-2\theta_j \lambda}) n| \geq \Delta\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_j^3 n}\})}. \quad (3.3)$$

Suppose  $\lambda = 1 + \sigma$  with  $\sigma \gg n^{-1/3}$  and  $1 \ll \alpha \ll (\theta_\lambda^3 n)^{1/2}$ . We take  $\frac{1 - \theta_\lambda}{10} \leq \beta \leq \frac{1 - \theta_\lambda}{2}$  so that  $(1 - \beta)^{a-1} = \theta_\lambda + \alpha(\theta_\lambda n)^{-1/2}$  for an integer  $a$ . Let

$$\Delta_j = 0.01 \alpha (\theta_j^3 n)^{1/2} \sum_{i=1}^j (1 - \beta)^{\frac{2j-i-a}{4}}.$$

Then, since  $(1-\beta)^{\frac{i}{4}}\theta_j^{-3/2} = (1-\beta)^{3/2}(1-\beta)^{-5j/4}$  increase as  $j$  increases, and  $\theta_a = \theta_\lambda + \alpha(\theta_\lambda n)^{-1/2} = (1+o(1))\theta_\lambda$ , we have

$$\alpha(1-\beta)^{\frac{i-a}{4}}(\theta_j^3 n)^{-1/2} \leq \alpha(\theta_a^3 n)^{-1/2} \ll 1. \quad (3.4)$$

and

$$\Delta_j = 0.01\alpha(\theta_j^3 n)^{1/2} \sum_{i=1}^j (1-\beta)^{\frac{2j-i-a}{4}} = 0.01\alpha(\theta_j^3 n)^{1/2} (1-\beta)^{\frac{j-a}{4}} \sum_{i=1}^j (1-\beta)^{\frac{i-j}{4}} \ll \theta_j^3 n$$

for all  $j = 1, \dots, a$ .

**Lemma 3.2** *For all  $\ell = 1, \dots, a$ , we have*

$$\Pr \left[ \exists j = 1, \dots, \ell \text{ s.t. } |N_j - 2\theta_j^2 \lambda (1 - \lambda e^{-2\theta_j \lambda}) n| > \Delta_j \right] \leq e^{-\Omega(\alpha^2 (1-\beta)^{\frac{\ell-a}{2}})},$$

and

$$\Pr \left[ \exists j = 1, \dots, \ell \text{ s.t. } |L_j - 2\theta_j \lambda (\theta_j - 1 + e^{-\theta_j \lambda}) n| > 2\Delta_j \right] \leq e^{-\Omega(\alpha^2 (1-\beta)^{\frac{\ell-a}{2}})}.$$

**Proof.** Let  $n_j = 2\theta_j^2 \lambda (1 - \lambda e^{-2\theta_j \lambda}) n$ ,  $b_j = (\theta_{j+1} \lambda)^2 e^{-2\theta_{j+1} \lambda} (1 - e^{-2\theta_j \lambda}) n$  and  $\alpha_j = (1-\beta)^{\frac{j-a}{4}} \alpha$ . Then  $\alpha_j \ll (\theta_j^3 n)^{1/2}$  by (3.4) and

$$n_{j+1} = (1-\beta)^2 n_j - 2b_j.$$

Since

$$\begin{aligned} \Pr \left[ \exists j = 1, \dots, \ell+1 \text{ s.t. } |N_j - n_j| > \Delta_j \right] &= \Pr \left[ \exists j = 1, \dots, \ell \text{ s.t. } |N_j - n_j| > \Delta_j \right] \\ &\quad + \Pr \left[ |N_{\ell+1} - n_{\ell+1}| > \Delta_{\ell+1} \right] \end{aligned}$$

and

$$\Pr \left[ |N_j - n_j| \leq \Delta_j \forall j \leq \ell, |N_{\ell+1} - n_{\ell+1}| > \Delta_{\ell+1} \right] \leq \Pr \left[ |N_\ell - n_\ell| \leq \Delta_\ell, |N_{\ell+1} - n_{\ell+1}| > \Delta_{\ell+1} \right],$$

it is enough by  $\sum_{j=1}^{\ell+1} e^{-\Omega(\alpha_j^2)} = e^{-\Omega(\alpha_{\ell+1}^2)}$  to show that

$$P_{\ell+1} := \Pr \left[ |N_\ell - n_\ell| \leq \Delta_\ell, |N_{\ell+1} - n_{\ell+1}| > \Delta_{\ell+1} \right] \leq e^{-\Omega(\alpha_{\ell+1}^2)}.$$

Notice that  $N_{\ell+1} = N_\ell - M_\ell - 2B_\ell$ ,  $n_{\ell+1} = (1-\beta)^2 n_\ell - 2b_\ell$  and

$$\begin{aligned} |N_{\ell+1} - n_{\ell+1}| &\leq |N_\ell - M_\ell - (1-\beta)^2 N_\ell| + (1-\beta)^2 |N_\ell - n_\ell| + |(1-\beta)^2 n_\ell - n_{\ell+1} - 2B_\ell| \\ &= |M_\ell - (1 - (1-\beta)^2) N_\ell| + (1-\beta)^2 |N_\ell - n_\ell| + 2|B_\ell - b_\ell|. \end{aligned}$$

As  $\Delta_{\ell+1} = (1-\beta)^2 \Delta_\ell + 0.01\alpha_{\ell+1}(\theta_{\ell+1}^3 n)^{1/2}$ , (3.2) and  $\alpha_j \ll (\theta_j^3 n)^{1/2}$  give

$$\begin{aligned} P_{\ell+1} &\leq \Pr \left[ |B_\ell - b_\ell| > \frac{1}{400} \alpha_{\ell+1} (\theta_{\ell+1}^3 n)^{1/2} \right] \\ &\quad + \Pr \left[ |M_\ell - (1 - (1-\beta)^2) N_\ell| > \frac{1}{200} \alpha_{\ell+1} (\theta_{\ell+1}^3 n)^{1/2}, |N_\ell - n_\ell| \leq \Delta_\ell \right] \\ &\leq e^{-\Omega(\alpha_{\ell+1}^2)} + \Pr \left[ |M_\ell - (1 - (1-\beta)^2) N_\ell| > \frac{1}{200} \alpha_{\ell+1} (\theta_{\ell+1}^3 n)^{1/2} \mid |N_\ell - n_\ell| \leq \Delta_\ell \right]. \end{aligned}$$

The desired bound follows, since (3.1) yields

$$\Pr \left[ |M_\ell - (1 - (1 - \beta)^2)N_\ell| > \frac{1}{200}\alpha_{\ell+1}(\theta_{\ell+1}^3 n)^{1/2} \middle| N_\ell \right] \leq e^{-\Omega(\alpha_{\ell+1}^2)}$$

for given  $N_\ell$  with  $|N_\ell - n_\ell| \leq \Delta_\ell \ll \theta_\ell^3 n$ .

The second inequality holds for (3.3) gives

$$\Pr \left[ \exists j = 1, \dots, \ell \text{ s.t. } |H_j - 2\theta_j \lambda (1 - e^{-\theta_j \lambda} - \theta_j \lambda e^{-2\theta_j \lambda}) n| \geq \Delta_j \right] \leq 2 \sum_{j=1}^{\ell} e^{-\Omega(\alpha_j^2)} = e^{-\Omega(\alpha_\ell^2)}.$$

□

**Proof of Main Lemma.** We first estimate the probability that all light clones disappear during phase  $j$ . Observe that the number of light clones is bounded by renewal random walk processes with negative drift: If the chosen light clone is matched to another light clone, then the number decreases by 2. If it is matched to a clone of a variable with type larger than or equal to  $(2, 2)$ , the number decreases by 1. If it is matched to a clone of a variable with type  $(1, b)$  or  $(b, 1)$ ,  $b \geq 2$ , then the number decreases, in expectation, by  $-1 + \frac{b}{b+1}$ . Thus, there is absolute constant  $h > 0$  such that the expected number of light clones is less than  $-h$ .

If all light clones disappear during phase  $j$ ,  $j = 1, \dots, a-1$ , then, either  $L_{j+1} \leq 0.1\alpha(\theta_{j+1}^3 n)^{1/2}$  or the renewal random walks with negative drift must reach beyond  $0.1\alpha(\theta_{j+1}^3 n)^{1/2}$ . Lemma 3.2 gives the probability of the former is  $e^{-\Omega(\alpha^2(1-\beta)^{\frac{\ell-a}{2}})}$ . For the latter, observe that the total number of walks is less than  $N_j$ , which is  $O(\theta_j^3 n)$  with probability  $e^{-\Omega(\alpha^2(1-\beta)^{\frac{j-a}{2}})}$ . We consider excursions that are segments of the renewal random walks between two consecutive visits to 0. The generalized Chernoff bound, or Corollary 2.4, with  $a_i, b_i, \xi, \delta = \Theta(1)$  yields that the probability of each excursion reaching beyond  $0.1\alpha(\theta_{j+1}^3 n)^{1/2}$  is at most

$$\sum_{m \geq 1} e^{-\Omega(\min\{\alpha(\theta_{j+1}^3 n)^{1/2} + hm, (\alpha(\theta_{j+1}^3 n)^{1/2} + hm)^2/m\})} \leq \sum_{m \geq 1} e^{-\Omega(\alpha(\theta_{j+1}^3 n)^{1/2}) - \Omega(m)} = e^{-\Omega(\alpha(\theta_{j+1}^3 n)^{1/2})}.$$

As there are at most  $N_j$  excursions such an excursion exists with probability at most

$$e^{-\Omega(\alpha^2(1-\beta)^{\frac{j-a}{2}})} + O\left(\theta_j^3 n e^{-\Omega(\alpha(\theta_{j+1}^3 n)^{1/2})}\right) \leq e^{-\Omega(\alpha^2(1-\beta)^{\frac{j-a}{2}})} + e^{-\Omega(\alpha^2(1-\beta)^{\frac{3(j-a)}{2}})} = e^{-\Omega(\alpha^2(1-\beta)^{\frac{j-a}{2}})}.$$

Therefore, there exists no light clones in a step of the  $j^{\text{th}}$  phase is at most  $e^{-\Omega(\alpha^2(1-\beta)^{\frac{\ell-a}{2}})}$ , which yields

$$\Pr[\Lambda_C \geq \theta_\lambda \lambda + \alpha(\theta_\lambda n)^{-1/2}] \leq e^{-\Omega(\alpha^2)},$$

replacing  $\alpha$  by  $\alpha/\lambda$ .

On the other hand, after  $(a-1)^{\text{th}}$  phase, there are at most  $O(\alpha(\theta_\lambda^3 n)^{1/2})$  light clones and  $\Omega(\theta_\lambda^3 n)$  unmatched clones of variables of type larger than  $(1, 1)$ , with probability  $1 - e^{-\Omega(\alpha^2)}$ . As the number of light clones has negative drift, no light clone exists after  $O(\alpha(\theta_\lambda^3 n)^{1/2}) = O(\frac{\alpha}{(\theta_\lambda^3 n)^{1/2}} \theta_\lambda^3 n)$  more clones are matched with probability  $1 - e^{-\Omega(\alpha^2)}$ . Thus, the cut-off value when no light clone exists for the first time cannot be smaller than  $(1 - O(\frac{\alpha}{(\theta_\lambda^3 n)^{1/2}}))\theta_\lambda \lambda = \theta_\lambda \lambda - O(\alpha(\theta_\lambda n)^{-1/2})$  with probability  $1 - e^{-\Omega(\alpha^2)}$  by the cut-off line lemma, or (3.1). Replacing  $\alpha$  by  $c\alpha$  for appropriate constant, we conclude that

$$\Pr[\Lambda_C \leq \theta_\lambda \lambda - \alpha(\theta_\lambda n)^{-1/2}] \leq e^{-\Omega(\alpha^2)}.$$

□

Let  $F_C^*(n, p)$  be the core of  $F_{PC}(n, p)$ . One may define  $C_{n,p}^*$ ,  $C_{n,p}^*(i, j)$ ,  $D_{n,p}^*(i, j)$  and  $M_{n,p}^*(i, j)$  for  $F_{PC}(n, p)$  as  $C_{n,p}$ ,  $C_{n,p}(i, j)$ ,  $D_{n,p}(i, j)$  and  $M_{n,p}(i, j)$  are defined for  $F(n, p)$ . We first estimate  $|D_{n,p}^*(i, j)|$  and  $M_{n,p}^*(i, j)$ . Notice that the upper and lower bounds for  $|D_{n,p}^*(i', j')|$ 's yield bounds for  $|C_{n,p}^*(i, j)|$  as

$$|C_{n,p}^*(i, j)| = |D_{n,p}^*(i, j)| - |D_{n,p}^*(i+1, j)| - |D_{n,p}^*(i, j+1)| + |D_{n,p}^*(i+1, j+1)|.$$

Let  $D_\alpha^\pm(i, j)$  be the sets of variables that are of type larger than or equal to  $(i, j)$  at  $\mu_\alpha^\pm := \theta_\lambda \lambda \pm \alpha(\theta_\lambda n)^{-1/2}$ , respectively, and let  $M_\alpha^\pm(i, j)$  be the number of clones of variables in  $D_\alpha^\pm(i, j)$  less than  $\mu_\alpha^\pm$ , respectively. Then, Lemma 3.1 gives

$$\Pr \left[ D_\alpha^-(i, j) \subseteq D_{n,p}^*(i, j) \subseteq D_\alpha^+(i, j) \text{ for all } i, j \right] = 1 - e^{-\Omega(\alpha^2)},$$

and

$$\Pr \left[ M_\alpha^-(i, j) \leq M_{n,p}^*(i, j) \leq M_\alpha^+(i, j) \text{ for all } i, j \right] = 1 - e^{-\Omega(\alpha^2)}.$$

Since  $|D_\alpha^\pm(i, j)|$  and  $M_\alpha^\pm(i, j)$  are the sums of i.i.d random variables and it is easy to check all the conditions of the generalized Chernoff bound with  $a_i, b_i = \Theta(\theta_\lambda^{i+j})$ ,  $\xi_0 = \delta = 1$ , we have

$$\Pr \left[ \left| |D_\alpha^\pm(i, j)| - Q_i(\mu_\alpha^\pm) Q_j(\mu_\alpha^\pm) n \right| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j} n}\})},$$

respectively, and

$$\Pr \left[ \left| M_\alpha^\pm(i, j) - \mu_\alpha^\pm \left( Q_{i-1}(\mu_\alpha^\pm) Q_j(\mu_\alpha^\pm) + Q_i(\mu_\alpha^\pm) Q_{j-1}(\mu_\alpha^\pm) \right) n \right| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j} n}\})},$$

respectively. Therefore,

$$\Pr \left[ |D_{n,p}^*(i, j)| - Q_i(\mu_\alpha^+) Q_j(\mu_\alpha^+) n \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j} n}\})} + e^{-\Omega(\alpha^2)},$$

$$\Pr \left[ |D_{n,p}^*(i, j)| - Q_i(\mu_\alpha^-) Q_j(\mu_\alpha^-) n \leq -\Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j} n}\})} + e^{-\Omega(\alpha^2)},$$

and

$$\Pr \left[ M_{n,p}^*(i, j) - \left( Q_{i-1}(\mu_\alpha^+) Q_j(\mu_\alpha^+) + Q_i(\mu_\alpha^+) Q_{j-1}(\mu_\alpha^+) \right) \mu_\alpha^+ n \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j} n}\})} + e^{-\Omega(\alpha^2)},$$

$$\Pr \left[ M_{n,p}^*(i, j) - \left( Q_{i-1}(\mu_\alpha^-) Q_j(\mu_\alpha^-) + Q_i(\mu_\alpha^-) Q_{j-1}(\mu_\alpha^-) \right) \mu_\alpha^- n \leq -\Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j} n}\})} + e^{-\Omega(\alpha^2)}.$$

We also have

$$\begin{aligned} \Pr[|D_{n,p}^*(i, j)| \geq \ell] &\leq e^{-\Omega(\alpha^2)} + \Pr[|D_{n,p}^+(i, j)| \geq \ell] \\ &\leq e^{-\Omega(\alpha^2)} + \binom{n}{\ell} (Q_i(\mu_\alpha^+) Q_j(\mu_\alpha^+))^\ell \\ &\leq e^{-\Omega(\alpha^2)} + \frac{(Q_i(\mu_\alpha^+) Q_j(\mu_\alpha^+) n)^\ell}{\ell!}. \end{aligned} \tag{3.5}$$

If  $(i, j)$  is fixed, it is easy to see that

$$Q_i(\mu_\alpha^\pm)Q_j(\mu_\alpha^\pm) = Q_i(\theta_\lambda \lambda)Q_j(\theta_\lambda \lambda) + O(\alpha\theta_\lambda^{i+j-1}(\theta_\lambda n)^{-1/2}),$$

and similarly

$$\mu_\alpha^\pm Q_i(\mu_\alpha^\pm)Q_j(\mu_\alpha^\pm) = \theta_\lambda \lambda Q_i(\theta_\lambda \lambda)Q_j(\theta_\lambda \lambda) + O(\alpha\theta_\lambda^{i+j}(\theta_\lambda n)^{-1/2}).$$

Replacing  $\alpha$  by  $c\alpha$  for an appropriate constant  $c > 0$  and taking  $\Delta = c\alpha\theta_\lambda^{i+j-1}(\theta_\lambda n)^{-1/2}n = c\alpha\theta_\lambda^{i+j-3}(\theta_\lambda^3 n)^{1/2}$ , we have, for  $Q_i = Q_i(\theta_\lambda \lambda)$ ,

$$\Pr \left[ \left| D_{n,p}^*(i, j) \right| - Q_i Q_j n \geq \alpha\theta_\lambda^{i+j-3}(\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\theta_\lambda^{i+j-3}\alpha^2)} + e^{-\Omega(\alpha^2)},$$

and

$$\Pr \left[ \left| M_{n,p}^*(i, j) - \theta_\lambda \lambda (Q_{i-1}Q_j + Q_i Q_{j-1})n \right| \geq \alpha\theta_\lambda^{i+j-3}(\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\theta_\lambda^{i+j-3}\alpha^2)} + e^{-\Omega(\alpha^2)}.$$

Thus, for  $F(n, p)$ , Theorem 1.1 gives

**Theorem 3.3** *Suppose  $p(2n - 1) = 1 + \sigma$  is uniformly bounded from above with  $\sigma \gg n^{-1/3}$ . Let  $\lambda = 1 + \sigma$ ,  $\Delta > 0$ ,  $1 \ll \alpha \ll (\theta_\lambda n)^{1/2}$ , and  $\mu_\alpha^\pm = \theta_\lambda \lambda \pm \alpha(\theta_\lambda n)^{-1/2}$ , respectively. Then,*

$$\Pr \left[ \left| D_{n,p}(i, j) \right| - Q_i(\mu_\alpha^+)Q_j(\mu_\alpha^+)n \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j}n}\})} + e^{-\Omega(\alpha^2)},$$

$$\Pr \left[ \left| D_{n,p}(i, j) \right| - Q_i(\mu_\alpha^-)Q_j(\mu_\alpha^-)n \leq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j}n}\})} + e^{-\Omega(\alpha^2)},$$

and

$$\Pr[|D_{n,p}(i, j)| \geq \ell] \leq O\left(\frac{((1 + \frac{\alpha}{(\theta_\lambda n)^{1/2}})\theta_\lambda \lambda)^{(i+j)\ell/2}}{(\ell!)^{1/2}}\right) + e^{-\Omega(\alpha^2)}.$$

We also have

$$\Pr \left[ M_{n,p}(i, j) - \mu_\alpha^+ (Q_{i-1}(\mu_\alpha^+)Q_j(\mu_\alpha^+) + Q_i(\mu_\alpha^+)Q_{j-1}(\mu_\alpha^+))n \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j}n}\})} + e^{-\Omega(\alpha^2)},$$

$$\Pr \left[ M_{n,p}(i, j) - \mu_\alpha^- (Q_{i-1}(\mu_\alpha^-)Q_j(\mu_\alpha^-) + Q_i(\mu_\alpha^-)Q_{j-1}(\mu_\alpha^-))n \leq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{\theta_\lambda^{i+j}n}\})} + e^{-\Omega(\alpha^2)}.$$

In particular, for fixed  $(i, j)$ ,

$$\Pr \left[ \left| D_{n,p}(i, j) \right| - Q_i Q_j n \geq \alpha\theta_\lambda^{i+j-3}(\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\theta_\lambda^{i+j-3}\alpha^2)} + e^{-\Omega(\alpha^2)},$$

and

$$\Pr \left[ \left| M_{n,p}(i, j) - \theta_\lambda \lambda (Q_{i-1}Q_j + Q_i Q_{j-1})n \right| \geq \alpha\theta_\lambda^{i+j-3}(\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\theta_\lambda^{i+j-3}\alpha^2)} + e^{-\Omega(\alpha^2)}.$$

□

Theorem 3.3 together with

$$|C_{n,p}(i, j)| = |D_{n,p}(i, j)| - |D_{n,p}(i + 1, j)| - |D_{n,p}(i, j + 1)| + |D_{n,p}(i + 1, j + 1)|.$$

implies that

$$\Pr \left[ \left| |C_{n,p}(i,j)| - P_i P_j n \right| \geq \alpha \theta_\lambda^{i+j-3} (\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\theta_\lambda^{i+j-3} \alpha^2)} + e^{-\Omega(\alpha^2)}.$$

Similarly, if  $i \geq j$ , then

$$|D_{n,p}(i,j) \cup D_{n,p}(j,i)| = |D_{n,p}(i,j)| + |D_{n,p}(j,i)| - |D_{n,p}(i,i)|$$

gives

$$\Pr \left[ \left| |D_{n,p}(i,j) \cup D_{n,p}(j,i)| - (2Q_i Q_j n - Q_i Q_i) n \right| \geq \alpha \theta_\lambda^{i+j-3} (\theta_\lambda^3 n)^{1/2} \right] \leq e^{-\Omega(\theta_\lambda^{i+j-3} \alpha^2)} + e^{-\Omega(\alpha^2)}.$$

The last two bounds of Theorem 1.4 are already in Theorem 3.3.

Furthermore, as  $C_{n,p} = D_{n,p}(1,1)$ ,  $K_{n,p} = D_{n,p}(2,1) \cup D_{n,p}(1,2)$ ,  $2|F_C(n,p)| = M_{n,p}(1,1)$ ,  $2|F_K(n,p)| = M_{n,p}(1,2) + M_{n,p}(2,1) - M_{n,p}(2,2)$  and  $Q_1(\theta_\lambda) = 1 - e^{-\theta_\lambda} = \theta_\lambda$ , Corollary 1.5 follows from Theorem 1.4.

Finally, the first two bounds in Corollary 1.6 follow from Theorem 1.4 since  $\theta_\lambda = \Theta(\sigma)$ . For the last bound, if  $\sigma^4 n \leq \varepsilon$  for a small positive constant  $\varepsilon$ , then Theorem 1.4 implies that all variables in the core are of types  $(1,1)$ ,  $(1,2)$  or  $(2,1)$ , with probability  $1 - O(\varepsilon^2)$ . If  $\sigma^4 n \geq \beta$  for a large constant  $\beta > 0$ , then Theorem 1.4 with  $\alpha = \beta^{-0.1} (\theta_\lambda^3 n)^{1/2}$  also gives

$$\Pr[M_{n,p}(2,2) - 2\theta_\lambda \lambda Q_1(\theta_\lambda \lambda) Q_2(\theta_\lambda \lambda) n \geq \beta^{-0.1} \theta_\lambda^4 n] \leq e^{-\Omega(\beta^{-0.2} \theta_\lambda^4 n)} \leq e^{-\Omega(\beta^{0.8})}.$$

Similar bounds hold for  $M_{n,p}(1,3)$  and  $M_{n,p}(3,1)$ . As  $\theta_\lambda = \Theta(\sigma)$  and  $\theta_\lambda \lambda Q_1(\theta_\lambda \lambda) Q_2(\theta_\lambda \lambda) = \Theta(\theta_\lambda^4)$ , the desired bound follows. If  $\varepsilon \leq \sigma^4 n \leq \beta$ , we simply use the bound for  $\lambda' = 1 + (\beta/n)^{1/4}$ , or  $p' = \frac{1+(\beta/n)^{1/4}}{2n-1}$ : Since  $\Pr[M_{n,p} \geq h] \leq \Pr[M_{n,p'} \geq h]$  and  $M_{n,p'} = O(\beta) = O((\beta/\varepsilon)\sigma^4 n)$  with probability  $1 - e^{-\Omega(\beta^{0.8})}$ , the bound follows.

## 4 Scaling Window: Proofs of Theorems 1.3, 1.7 and 1.8

Suppose all  $C_{n,p}(i,j)$ 's for  $(i,j) \geq (1,1)$  are given. The first thing we need to establish is that all 2-SAT formulae with the same  $C_{n,p}(i,j)$ 's are equally likely to be the core of  $F(n,p)$ . More generally, it is not hard to show the following lemma.

**Theorem 4.1 (Restated)** *Suppose two formulae have the same number of clauses on the same number of underlying variables, and all underlying variables are of type at least  $(1,1)$ . Then the two are equally likely to be the core of  $F(n,p)$ .*

**Proof.** Let  $F_1$  and  $F_2$  be the two formulae. After an appropriate permutation, we may assume that the two formulae have the same set of underlying variables. Then, a formula having  $F_1$  as its core can be mapped to the formula obtained by replacing clauses in  $F_1$  with clauses of  $F_2$ . It is easy to see that the core of the formula obtained this way is  $F_2$ . It is also clear that the map is one-to-one and onto. Furthermore, two formulae mapped each other have the same number of clauses, which means that the random formula  $F(n,p)$  is equally likely to be one of the two formulae.  $\square$

We now consider the configuration model for given  $C_{n,p}(i,j)$ : Similar to the Poisson cloning model, take  $i$  clones of  $x$  and  $j$  clones of  $\bar{x}$  for each variable  $x \in C_{n,p}(i,j)$ . The uniform random perfect matching on all clones is called the random configuration. The random configuration

then yields a 2-SAT multiformula after contractions. The event that the multiformula has neither loops nor multiple clauses is called *SIMPLE* or *SIM*. Conditioned on *SIM*, the random 2-SAT formula has the uniform distribution among all 2-SAT formulae with the same  $C_{n,p}(i, j)$ 's. This is not difficult to see as the number of perfect matchings that yield a fixed 2-SAT formula is  $\prod_{(i,j)} (i!j!)^{|C_{n,p}(i,j)|}$ . It is known that the probability of *SIM* is uniformly bounded below from 0, especially, for any event  $A$  in the uniform model, or equivalently in the configuration model,

$$\Pr[A] = \Pr^*[A|SIM] \leq \Pr^*[SIM]^{-1} \Pr^*[A] = O(\Pr^*[A]), \quad (4.1)$$

where the probability  $\Pr^*$  is taken over the random configuration without any condition. Hence, as far as the constant factor is not concerned, it is enough to bound the desired probabilities in the configuration without any condition. To clarify terminology, we recall that the configuration model is obtained from the random configuration by conditioning *SIM*. For an event  $A$  depending on the random configuration only, such as the event that the  $i^{\text{th}}$  clone of  $y$  and the  $j^{\text{th}}$  clone of  $z$  are matched,  $\Pr[A]$  may not be well-defined, but  $\Pr^*[A]$  or  $\Pr^*[A|SIM]$  may be still considered.

We may be able to estimate the probability of *SIM* in the case that all but few clones are clones of type  $(1, 1)$  literals: Suppose all  $N$  but  $o(N^{1/2})$  clones are clones of type  $(1, 1)$  literals. First, with probability  $1 - o(1)$ , no pair of clones that are not clones of type  $(1, 1)$  literals is matched. Thus, the multiformula is not simple mainly because two clones of type  $(1, 1)$  variable  $x$  and its negation are matched. Let  $A_x$  be such an event. Then, for the set  $U$  of all type  $(1, 1)$  variables,

$$\sum_{\ell=0}^{2i+1} (-1)^\ell \sum_{\substack{W \subseteq U \\ |W|=\ell}} \Pr^* \left[ \bigcap_{x \in W} A_x \right] \leq \Pr^* \left[ \overline{\bigcup_{x \in W} A_x} \right] \leq \sum_{\ell=0}^{2i} (-1)^\ell \sum_{\substack{W \subseteq U \\ |W|=\ell}} \Pr^* \left[ \bigcap_{x \in W} A_x \right],$$

for all  $i \geq 0$ . For  $\ell = o(N^{1/2})$  and  $|W| = \ell$ ,

$$\binom{|U|}{\ell} = \binom{N/2 - o(N^{1/2})}{\ell} = \frac{(1 + O(\frac{\ell(\ell+N^{1/2})}{N}))(\frac{N}{2})^\ell}{\ell!}$$

and

$$\Pr^*[\bigcap_{x \in W} A_x] = \frac{(N - 2\ell - 1)!!}{(N - 1)!!} = \frac{1 + O(\frac{\ell^2}{N})}{N^\ell}.$$

Therefore,

$$\Pr^* \left[ \overline{\bigcup_{x \in W} A_x} \right] = (1 + o(1))e^{-1/2}, \quad \text{and} \quad \Pr^*[SIM] = (1 + o(1))e^{-1/2} \quad (4.2)$$

We are now ready to prove Theorems 1.7 and 1.8.

**Proof of Theorem 1.7** We may generate  $F(n, p)$  with  $\lambda_p := p(2n - 1) = 1 - \sigma$  by first taking  $F(n, q)$  with  $\lambda_q = 1 + n^{-1/3} \log(\sigma^3 n)$  and then independently selecting each clause of  $F(n, q)$  with probability  $p/q = \frac{1-\sigma}{1+n^{-1/3} \log(\sigma^3 n)} = 1 - \sigma + o(\sigma)$ .

Applying Corollary 1.5 for  $\lambda_q$  and  $\alpha = \log(\sigma^3 n)$  and using  $\theta_q := \theta_{\lambda_q} = 2n^{-1/3} \log(\sigma^3 n) + O(n^{-2/3} \log^2(\sigma^3 n))$ , we have

$$|C_{n,q}| = \theta_q^2 q^2 n + O(\theta_q^3 n) + O((\theta_q n)^{1/2} \log(\sigma^3 n)) = (4 + o(1))n^{1/3} \log^2(\sigma^3 n),$$

$$|K_{n,q}| = \theta_q^3 \lambda^3 n + O(\theta_q^4 n) + O((\theta_q^3 n)^{1/2} \log(\sigma^3 n)) = (8 + o(1)) \log^3(\sigma^3 n)$$



and

$$|F_C(n, q)| = \theta_q^2 \lambda^2 n + O(\theta_q^3 n) + O((\theta_q n)^{1/2} \log(\sigma^3 n)) = (4 + o(1)) n^{1/3} \log^2(\sigma^3 n),$$

$$|F_K(n, q)| = \frac{3}{2} \theta_q^3 \lambda^3 n + O(\theta_q^4 n) + O((\theta_q^3 n)^{1/2} \log(\sigma^3 n)) = (12 + o(1)) \log^3(\sigma^3 n),$$

and for  $K_{n,q}(1, 2) := C_{n,q}(1, 2) \cup C_{n,q}(2, 1)$

$$|K_{n,q}(1, 2)| = (8 + o(1)) \log^3(\sigma^3 n)$$

with probability  $1 - e^{-\Omega(\log^2(\sigma^3 n))}$ . Theorem 3.3 with the same  $\alpha$  also gives

$$|D_{n,q}(i, j)| = 0 \quad \text{if } i + j \geq 7 \quad (4.3)$$

with probability  $1 - n^{-7/6+o(1)}$ .

Suppose  $C_{n,q}(i, j)$ 's are given with  $C_{n,q} := \cup_{(i,j) \geq (1,1)} C_{n,q}(i, j)$ ,  $K_{n,q} := \cup_{(i,j) > (1,1)} C_{n,q}(i, j)$ ,  $D_{n,q}(i, j) := \cup_{(i',j') > (i,j)} C_{n,q}(i', j')$ , and  $K_{n,q}(1, 2) := C_{n,q}(1, 2) \cup C_{n,q}(2, 1)$  satisfying the above conditions, and the number  $M_C$  and  $M_K$  of clones of variables in  $C_{n,q}$  and  $K_{n,q}$ , respectively, satisfy

$$M_C = \sum_{(i,j) \geq (1,1)} (i + j) |C_{n,q}(i, j)| = (8 + o(1)) n^{1/3} \log^2(\sigma^3 n),$$

and

$$M_K = \sum_{(i,j) > (1,1)} (i + j) |C_{n,q}(i, j)| = (24 + o(1)) \log^3(\sigma^3 n).$$

In the corresponding random configuration for given  $C_{n,q}(i, j)$ 's, two clones  $y, z$  of variables in  $K_{n,q}$  may yield a clause after resolutions of variables in  $C_{n,q}(1, 1)$ . This occurs if and only if there are  $x_1, \dots, x_\ell \in C_{n,q}(1, 1)$  such that  $\{w_0 := y, w_1\}, \{\bar{w}_1, w_2\}, \dots, \{\bar{w}_{\ell-1}, w_\ell\}, \{\bar{w}_\ell, w_{\ell+1} := z\}$  are edges in the random configuration, where  $w_i, \bar{w}_i$  are the two clones of  $x_i$  and  $\bar{x}_i$  (not necessarily respectively), including the case  $\ell = 0$ . If this event occurs, we say that the length  $\ell(y, z)$  of  $y, z$  is  $\ell + 1$  and the edges  $\{w_i, w_{i+1}\}$  (resp. the corresponding clauses after contractions) are called intermediate edges (resp. clauses) of the pair. The length  $\ell(y, z)$  is infinity if no such  $x_i$ 's exist. It is easy to see that  $\Pr^*[\ell(y, z) = 1] = \frac{1}{M_C - 1}$ . Similarly,

$$\Pr^*[\ell(y, z) = 2] = \left(1 - \frac{M_K - 1}{M_C - 1}\right) \frac{1}{M_C - 3},$$

and, in general,

$$\Pr^*[\ell(y, z) = \ell] = \left(1 - \frac{M_K - 1}{M_C - 1}\right) \left(1 - \frac{M_K - 1}{M_C - 3}\right) \cdots \left(1 - \frac{M_K - 1}{M_C - 2\ell + 3}\right) \frac{1}{M_C - 2\ell + 1}.$$

For  $i = 1, \dots, 4$  and  $\ell_1, \dots, \ell_i \leq \frac{10}{\sigma} \log(\sigma^3 n) \ll n^{1/3}$ , the same argument also gives

$$\begin{aligned} \Pr^*[\ell(y_j, z_j) = \ell_j, \quad j = 1, \dots, i] &= \left(1 - \frac{(1 + o(1))M_K}{M_C}\right)^{\ell_1 + \dots + \ell_i} \left(\frac{1 + o(1)}{M_C}\right)^i \\ &= \left(\frac{1 + o(1)}{8n^{1/3} \log^2(\sigma^3 n)}\right)^i. \end{aligned} \quad (4.4)$$

Notice that a pair  $y, z$  of clones of variables in  $K_{n,q}$  yields the corresponding clause in the kernel  $F_K(n, p)$  only if  $\ell(y, z) < \infty$  and all the  $\ell(y, z)$  intermediate clauses of the pair are in  $F(n, p)$ . Such

an event occurs with probability

$$\begin{aligned} \sum_{\ell \geq 1} (p/q)^\ell \Pr^*[\ell(y, z) = \ell | SIM] &= O\left( \sum_{\ell=1}^{\frac{10}{\sigma} \log(\sigma^3 n)} (1-\sigma)^\ell \Pr^*[\ell(y, z) = \ell] + (1-\sigma)^{\frac{10}{\sigma} \log(\sigma^3 n)} \right) \\ &= O\left( \sum_{\ell=1}^{\frac{10}{\sigma} \log(\sigma^3 n)} (1-\sigma)^\ell \Pr^*[\ell(y, z) = \ell] \right) + O((\sigma^3 n)^{-10}). \end{aligned}$$

Similarly, if the kernel  $F_K(n, p)$  of  $F(n, p)$  has  $i$  or more clauses,  $i = 1, \dots, 4$ , then there must be  $i$  distinct pairs  $\{y_j, z_j\}$  of clones of variables in  $K_{n,q}$  such that  $\ell(y_j, z_j) < \infty$  and all the  $\ell(y_j, z_j)$  intermediate clauses of each pair are in  $F(n, p)$ ,  $j = 1, \dots, i$ . The probability of such event is at most

$$O\left( \binom{M_K}{8} \sum_{\ell_1, \dots, \ell_i \geq 1} (p/q)^{\ell_1 + \dots + \ell_i} \Pr^*[\ell(y_j, z_j) = \ell_j, j = 1, \dots, i | SIM] \right)$$

for fixed  $i$  distinct pairs  $\{y_i, z_i\}$  of clones of variables in  $K_{n,p}$ ,  $i = 1, \dots, 4$ . Since  $M_K = O(\log^3(\sigma^3 n))$  and

$$\begin{aligned} P_i &:= \sum_{\ell_1, \dots, \ell_i \geq 1} (p/q)^{\ell_1 + \dots + \ell_i} \Pr^*[\ell(y_j, z_j) = \ell_j, j = 1, \dots, i] \\ &\leq \sum_{\ell_1, \dots, \ell_i \geq 1} (1-\sigma)^{\ell_1 + \dots + \ell_i} \Pr^*[\ell(y_j, z_j) = \ell_j, j = 1, \dots, i] + (1-\sigma)^{\frac{10}{\sigma} \log(\sigma^3 n)} \\ &\leq \left( \frac{1 + o(1)}{8\sigma n^{1/3} \log^2(\sigma^3 n)} \right)^i + (\sigma^3 n)^{-10}, \end{aligned}$$

that

$$\Pr[|F_K(n, p)| \geq 4] \leq (\sigma^3 n)^{-4/3+o(1)}.$$

Thus, it is enough to estimate the probability of  $|F_K(n, p)| = 2, 3$  since kernels must have two or more clauses. For two variables  $w, x$  in  $K_{n,q}$ , let  $A_{wx}$  be the event of  $K_{n,p} = \{w, x\}$ ,  $|F_K(n, p)| = 3$  and no variables in  $K_{n,q}$  are in intermediate clauses, and let  $B_{wx}$  be the event that, in addition to  $A_{wx}$ , each of the three clauses in  $F_K(n, p)$  has at least 1 but not more than  $\frac{10}{\sigma} \log(\sigma^3 n)$  intermediate clauses. Clearly, for  $K_{n,q}(1, 2) := C_{n,q}(1, 2) \cup C_{n,q}(2, 1)$ ,

$$\Pr[|K_{n,p}| = 2, |F_K(n, p)| = 3] \geq \Pr \left[ \bigcup_{\substack{w, x \in K_{n,q}(1, 2) \\ w \neq x}} B_{wx} \right]. \quad (4.5)$$

For an upper bound, if  $|K_{n,p}| = 2, |F_K(n, p)| = 3$  but  $\bigcup_{\substack{w, x \in K_{n,q} \\ w \neq x}} A_{wx}$  does not occur, then at least 4 distinct pairs of clones of variables in  $K_{n,q}$  must have finite length and all corresponding intermediate clauses of them must be in  $F(n, p)$ . This probability is at most  $O(\binom{M_K}{8} P_4) = (\sigma^3 n)^{-4/3+o(1)}$ . The probability of  $\bigcup_{\{w, x\} \not\subseteq K_{n,q}(1, 2)} A_{wx}$  may be bounded by

$$\Pr \left[ \bigcup_{\{w, x\} \not\subseteq K_{n,q}(1, 2)} A_{wx} \right] = O(|K_{n,q}|(|K_{n,q}| - |K_{n,q}(1, 2)|)P_3) = o(\log^6(\sigma^3 n)P_3) = o((\sigma^3 n)^{-1}),$$

for  $|K_{n,q}| - |K_{n,q}(1, 2)| = o(\log^3(\sigma^3 n))$ . Finally,

$$\sum_{\substack{wx \in K_{n,q}(1, 2) \\ w \neq x}} \Pr[A_{wx} \setminus B_{wx}] = O\left(|K_{n,q}|^2 \frac{P_2}{M_C}\right) + O\left(|K_{n,q}|^2 (1-\sigma)^{\frac{10}{\sigma} \log(\sigma^3 n)}\right) = O((\sigma^2 n)^{-1}).$$

All together, we have

$$\Pr \left[ |K_{n,p}| = 2, |F_K(n, p)| = 3 \right] = \Pr \left[ \bigcup_{\substack{wx \in K_{n,q}(1,2) \\ w \neq x}} B_{wx} \right] + o((\sigma^3 n)^{-1}).$$

Moreover, as

$$0 \leq \sum_{\substack{wx \in K_{n,q}(1,2) \\ w \neq x}} \Pr[B_{wx}] - \Pr \left[ \bigcup_{\substack{wx \in K_{n,q}(1,2) \\ w \neq x}} B_{wx} \right] \leq \sum_{\substack{w,x,w',x' \in K_{n,q}(1,2) \\ w \neq x, w' \neq x', \{w,x\} \neq \{w',x'\}}} \Pr[B_{wx} \cap B_{w'x'}]$$

and

$$\sum_{\substack{w,x,w',x' \in K_{n,q}(1,2) \\ w \neq x, w' \neq x', \{w,x\} \neq \{w',x'\}}} \Pr[B_{wx} \cap B_{w'x'}] = O(|K_{n,q}|^4 P_4) = o((\sigma^3 n)^{-1}),$$

we deduce that

$$\Pr \left[ |K_{n,p}| = 2, |F_K(n, p)| = 3 \right] = \sum_{\substack{wx \in K_{n,q}(1,2) \\ w \neq x}} \Pr[B_{wx}] + o((\sigma^3 n)^{-1}).$$

To estimate  $\Pr[B_{wx}]$ , we may assume that both of  $w$  and  $x$  are of type  $(2, 1)$ , after exchanging the roles  $w, x$  with  $\bar{w}, \bar{x}$  if needed. In the random configuration, there are  $5 \cdot 3 \cdot 1$  ways to match the 6 clones of  $w$  and  $x$ . In each case  $i = 1, \dots, 15$ , let  $B_i(\ell_1, \ell_2, \ell_3)$  be the event that there are  $\ell_1, \ell_2, \ell_3$  intermediate variables in  $K_{n,q}$  for the three matches. Then

$$\Pr[B_{wx}] = \sum_{i=1}^{15} \frac{10}{\sigma} \log(\sigma^3 n) \sum_{\ell_1, \ell_2, \ell_3 \geq 2} (1 - \sigma - o(\sigma))^{\ell_1 + \ell_2 + \ell_3} \Pr^*[B_i(\ell_1, \ell_2, \ell_3) | SIM].$$

For  $\ell_1, \ell_2, \ell_3$  in the above range, (4.2) and (4.3) give

$$\begin{aligned} \Pr^*[B_i(\ell_1, \ell_2, \ell_3) | SIM] &= \frac{\Pr^*[B_i(\ell_1, \ell_2, \ell_3) \cap SIM]}{\Pr^*[SIM]} \\ &= \left( \frac{1 + o(1)}{8n^{1/3} \log^2(\sigma^3 n)} \right)^3 \frac{\Pr^*[SIM']}{\Pr^*[SIM]} \\ &= \left( \frac{1 + o(1)}{8n^{1/3} \log^2(\sigma^3 n)} \right)^3, \end{aligned}$$

where  $SIM'$  is the event that the random configuration on the  $M_C - 6 - 2(\ell_1 + \ell_2 + \ell_3)$  clones is simple. Hence

$$\Pr[B_{wx}] = \frac{15 + o(1)}{8^3 \sigma^3 n \log^6(\sigma^3 n)},$$

and

$$\Pr \left[ |K_{n,p}| = 2, |F_K(n, p)| = 3 \right] = \binom{|K_{n,q}(1,2)|}{2} \frac{15 + o(1)}{8^3 \sigma^3 n \log^6(\sigma^3 n)} + o((\sigma^3 n)^{-1}) = \frac{15 + o(1)}{16 \sigma^3 n}.$$

If  $|K_{n,p}| = 1$ , then there are at least two clauses in  $F_K(n, p)$ . Appealing directly to  $F(n, p)$  with  $p = \frac{1-\sigma}{2n-1}$ ,

$$\Pr[|K_{n,p}| = 1] = O\left(n \sum_{\ell_1, \ell_2 \geq 1} \binom{n}{\ell_1 - 1} 2^{\ell_1 - 1} (\ell_1 - 1)! \binom{n}{\ell_2 - 1} 2^{\ell_2 - 1} (\ell_2 - 1)! \left(\frac{1 - \sigma}{2n - 1}\right)^{\ell_1 + \ell_2}\right) = O\left(\frac{1}{\sigma^2 n}\right),$$

where  $\ell_1$  and  $\ell_2$  represent the numbers of intermediate clauses.

When the event  $B_{wx}$  occurs for variables  $w, x$  of type  $(2, 1)$ , the only way (out of the 15 ways) it directly makes the formula unsatisfiable is the case that the two clones of each of  $w$  and  $x$  are matched and the two clones of  $\bar{w}$  and  $\bar{x}$  are matched. Therefore, the same argument yields

$$\Pr[F(n, p) \text{ is UNSAT}] = \frac{1 + o(1)}{16\sigma^3 n}.$$

□

**Proof of Theorem 1.8** At  $\Lambda_S$ ,  $F_K(n, p)$  has  $\Omega(\theta_\lambda^3 n)$  variables  $1 - e^{-\Omega(\theta_\lambda^3 n)}$  by Theorem 3.3 and the convention (see Section 1). Exchanging the roles of  $x$  and  $\bar{x}$ , if necessary, we may assume that the number of  $x$ -clones is at least as large as the number  $\bar{x}$ -clones for all variables  $x \in K_{n,p}$ . For the lower bound, let  $Y$  be the set of clones of  $x$ 's and  $Z$  be the set of clones of  $\bar{x}$ 's. Then

$$|Y| \geq |Z|.$$

We now consider the event that all clones in  $Z$  are matched to clones in  $Y$ , in which case,  $(1, \dots, 1)$  is a satisfying assignment. The probability of the event is

$$\frac{|Y|}{|Y| + |Z| - 1} \frac{|Y| - 1}{|Y| + |Z| - 3} \cdots \frac{|Y| - |Z| + 1}{|Y| - |Z| + 1} \geq 2^{-|Z|} \geq e^{-O(\theta_\lambda^3 n)} = e^{-O(\sigma^3 n)}.$$

For the upper bound, we may assume  $\sigma \leq 0.01$ . Since the probability decreases as  $\sigma$  increase, once the probability is at most  $e^{-\Omega(\sigma^3 n)}$  for  $\sigma = 0.01$ , the probability is at most  $e^{-\Omega(n)}$  for larger  $\sigma$ 's. Corollary 1.5 implies that, with probability  $1 - e^{-\Omega(\theta_\lambda^3 n)}$ ,

$$|K_{n,p}| \geq 0.99\theta_\lambda^3 n \quad \text{and} \quad M_{n,p}(2, 2) + M_{n,p}(1, 3) + M_{n,p}(3, 1) \leq 0.01\theta_\lambda^3 n. \quad (4.6)$$

It is enough to show the desired bound in the random configuration satisfying (4.6) as  $\Pr^*[SIM] = \Omega(1)$ . We first take the following procedure to make the problem simpler. Remove all the clauses in the kernel  $F_K(n, p)$  containing any variable not in  $K_{n,p}(1, 2)$  and its negation. (Recall  $K_{n,p}(1, 2) = C_{n,p}(1, 2) \cup C_{n,p}(2, 1)$ .) Then (4.6) implies that there are at most  $0.01\theta_\lambda^3 n$  such clauses. Furthermore, as at most one variable in  $K_{n,p}(1, 2)$  is affected by one such clause, there are at most  $0.02\theta_\lambda^3 n$  pure clones can be created. We now apply PLA: Each time a pure clone is matched, the number of pure clones decreases by 2 if it is matched to another pure clone. If it is matched to a non-pure clone, two clones become pure with probability  $1/3$ , and a variable becomes of type  $(1, 1)$  with the other probability. This is so since all non-pure variables are of type  $(1, 2)$  or  $(2, 1)$ . Hence, after each step, the number of pure clones decreases by at least  $1/3$  in expectation, and increases by no more than 1 at any case. The generalized Chernoff bound implies that no pure clone is left within  $0.07\theta_\lambda^3 n$  steps, with probability  $1 - e^{-\Omega(\theta_\lambda^3 n)}$ . Therefore, there are at least  $0.9\theta_\lambda^3 n$  variables of type  $(1, 2)$  or  $(2, 1)$  remain after PLA stops, with probability  $1 - e^{-\Omega(\theta_\lambda^3 n)}$ . The desired bound may be obtained from the next lemma.

**Lemma 4.2** *Let  $F(b)$  be the (multi)formula yielded by the random configuration on  $b$  variables of type  $(1, 2)$  or  $(2, 1)$ . Then*

$$\Pr[F(b) \text{ is SAT}] = o(e^{-0.02b}),$$

as  $b \rightarrow \infty$ .

**Proof.** After exchanging the roles of  $x$  and  $\bar{x}$  as needed, we may assume that all  $b$  variables are of type  $(2, 1)$ . Notice that an assignment for the  $b$  variables may be regarded as a  $0, 1$  vector of length  $b$ . That is, the  $i^{\text{th}}$  coordinate of it tells the truth value of the  $i^{\text{th}}$  variable, say  $x_i$ . Suppose an assignment has exactly  $tb$  0's. Then, there are  $2tb + (1 - t)b$  clones whose truth values are set to be 0. These clones are called negative. The other clones are set to be 1 and will be called positive. The assignment is a satisfying assignment if and only if there is no edge connecting two negative clones. We call such an edge *bad*. A clause corresponding a bad edge is also called *bad*.

If  $F := F(b)$  is satisfiable, then there are assignments that yield no bad clause. Among those assignments, we may take one with maximum number of 1's. Such an assignment is called maximal. Suppose an satisfying assignment  $s = (s_i)$  is maximal. Then, for a variable  $x_i$  with  $s_i = 0$ , the only clone of  $\bar{x}_i$ , which is a positive clone (with respect to  $s$ ), must be connected to a negative clone. Otherwise, the assignment  $s^*$  obtained from  $s$  by changing the value of  $s_i$  to 1 is another satisfying assignment, which implies that  $s$  is not maximal.

Summarizing, we have the followings. Provided  $s$  has  $tb$  0's, the number  $N$  of negative clones is  $2tb + (1 - t)b = (1 + t)b$  and the number  $M$  of positive clones is  $2(1 - t)b + tb = (2 - t)b$ . If  $s$  is a maximal satisfying assignment, then there is no bad clause and the positive clone of each  $\bar{x}_i$  with  $s_i = 0$  must be matched to a negative clone. The number  $L$  of positive clones of  $\bar{x}_i$  with  $s_i = 0$  is  $tb$ . Since all  $N$  negative clones must be matched to positive clones, and the  $L$  positive clones mentioned above must be matched to negative clones, and the number of perfect matchings on  $m$  clones for even  $m$  is

$$(m - 1)!! = \frac{m!}{2^{m/2}(m/2)!},$$

we have that

$$\begin{aligned} P(s) &:= \Pr[s \text{ is a maximal satisfying assignment}] \\ &\leq \frac{\binom{M-L}{N-L} N! (M - N - 1)!!}{(M + N - 1)!!} \end{aligned}$$

where

$$N = (1 + t)b, \quad M = (2 - t)b, \quad L = tb,$$

(provided  $s$  has  $tb$  0's). Using Stirling formula, we have

$$\begin{aligned} P(s) &\leq b \exp \left( 2(1 - t)bH\left(\frac{1}{2(1-t)}\right) + N \ln N + \frac{M - N}{2} \ln(M - N) - \frac{M + N}{2} \ln(M + N) \right) \\ &= b \exp \left( 2(1 - t)bH\left(\frac{1}{2(1-t)}\right) + (1 + t)b \ln \frac{1 + t}{3} + \frac{(1 - 2t)b}{2} \ln \frac{1 - 2t}{3} \right). \end{aligned}$$

Finally, by  $\binom{b}{tb} \leq e^{bH(t)}$  and

$$\max_{0 \leq t \leq 1/2} \left\{ H(t) + 2(1 - t)H\left(\frac{1}{2(1-t)}\right) + (1 + t) \ln \frac{1 + t}{3} + \frac{1 - 2t}{2} \ln \frac{1 - 2t}{3} \right\} < -0.02,$$

we have

$$\Pr[F \text{ is SAT}] \leq \sum_{t=0}^b \binom{b}{tb} P(s) = o(e^{-0.02b}).$$

□

## References

- [1] D. Achlioptas. Setting 2 variables at a time yields a new lower bound for random 3-SAT (extended abstract), *Proc. 32nd ACM Symposium on Theory of Computing*, 28–37 (2000).
- [2] D. Achlioptas. Lower Bounds for Random 3-SAT via Differential Equations, *Theoretical Computer Science*, **265**, 159–185 (2001).
- [3] D. Achlioptas and G.B. Sorkin. Optimal myopic algorithms for random 3-SAT, *Proc. 41st Symposium on the Foundations of Computer Science*, 590–600 (2000).
- [4] D. Achlioptas, L. Kirousis, E. Kranakis, and D. Krizanc. Rigorous Results for  $(2+p)$ -SAT, *Theoretical Computer Science*, **265**, 109–129 (2001).
- [5] A. Békéssy, P. Békéssy and J. Komlós, Asymptotic enumeration of regular matrices, *Studia Sci. Math. Hungar.*, **7** (1972), 343–353.
- [6] E. Bender and R. Canfield, The asymptotic number of labeled graphs with given degree sequences, *J. Combinatorial Theory Ser. A* **24** (1978), 296–307.
- [7] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *European J. Combin.* **1** (1980), 311–316.
- [8] B. Bollobás, C. Borgs, J. Chayes, J. H. Kim, D. Wilson, The Scaling Window of the 2-SAT Transition, *Random Structures and Algorithms*, **18** (2001), 201–256.
- [9] E.A. Bender, E.R. Canfield and B.D. McKay. The asymptotic number of labeled connected graphs with a given number of vertices and edges, *Rand. Struc. Alg.* **1**, 127–169 (1990).
- [10] A. Broder, A. Frieze and E. Upfal. On the satisfiability and maximum satisfiability of random 3-CNF formulas, *Proc. 4th ACM-SIAM Symposium on Discrete Algorithms*, 322–330 (1993).
- [11] M.T. Chao and J. Franco. Probabilistic analysis of a generalization of the unit-clause literal selection heuristics for the  $k$  satisfiable problem, *Information Science* **51**, 289–314 (1990).
- [12] V. Chvátal and B. Reed. Mick gets some (the odds are on his side), *Proc. 33rd Symposium on the Foundations of Computer Science*, 620–627 (1992).
- [13] S.A. Cook. The complexity of theorem-proving procedures, *Proc. 3rd ACM Symposium on Theory of Computing*, 151–158 (1971).
- [14] O. Dubois and Y. Boufkhad. A general upper bound for the satisfiability threshold of random  $k$ -SAT formulas, *J. Algorithms* **24**, 395–420 (1997).
- [15] O. Dubois, Y. Boufkhad, and J. Mandler. Typical random 3-SAT formulae and the satisfiability threshold. Research announcement at ICTP, Sept. 1999. Two-page abstract appears in *Proc. 11th ACM-SIAM Symposium on Discrete Algorithms*, 126–127 (2000).
- [16] A. El Maftouhi and W. Fernandez de la Vega. On random 3-sat. *Combin. Probab. Comput.* **4**, 189–195 (1995).
- [17] E. Friedgut, with appendix by J. Bourgain. Sharp thresholds of graph properties, and the  $k$ -sat problem, *J. Amer. Math. Soc.* **12**, 1017–1054 (1999).

- [18] W. Fernandez de la Vega. On random 2-SAT (revised version), preprint (1998).
- [19] J. Franco and M. Paull. Probabilistic analysis of the Davis-Putnam procedure for solving the satisfiability problem, *Discrete Applied Mathematics* **5**, 77–87 (1983).
- [20] A. Frieze and S. Suen. Analysis of two simple heuristics for a random instance of  $K$ -SAT, *J. Algorithms* **20**, 312–335 (1996).
- [21] A. Goerdt. A threshold for unsatisfiability, *J. Computer and System Sciences* **53**, 469–486 (1996).
- [22] S. Janson, Y.C. Stamatiou, and M. Vamvakari. Bounding the unsatisfiability threshold of random 3-SAT, *Rand. Struc. Alg.* **17**, 103–116 (2000).
- [23] A. Kamath, R. Motwani, K. Palem and P. Spirakis. Tail bounds for occupancy and the satisfiability threshold conjecture, *Rand. Struc. Alg.* **7**, 59–89 (1995).
- [24] J.-H. Kim, Poisson cloning model for random graph, manuscript, in <http://research.microsoft.com/theory/jehkim/>.
- [25] J.-H. Kim, Poisson cloning model for random digraph, in preparation.
- [26] L. Kirousis, E. Kranakis, D. Krizanc, and Y. Stamatiou, Approximating the unsatisfiability threshold of random formulas, *Random Structures and Algorithms*, **12**, 27–38 (1998).
- [27] A. Kaporis, L. Kirousis, Y. Stamatiou, M. Vamvakari and M. Zito. *The unsatisfiability threshold revisited*, submitted.
- [28] M. Mitzenmacher. Tight Thresholds for the Pure Literal Rule, SRC Technical Note 1997-011.
- [29] N. Wormald. Some Problems in the Enumeration of Labelled Graphs PhD thesis, University of Newcastle, 1978.
- [30] N. Wormald. Differential equations for random processes and random graphs, *Ann. Appl. Prob.* **5**, 1217–1235 (1995).
- [31] M. Zito. Randomised techniques in combinatorial algorithms, Ph.D thesis, Dept. of Comp. Sci., Univ. of Warwick, 1999.